Sheet #2

1 Determine the volume V of a region defined in a cylindrical coordinate system as $1m \le r \le 2m$, $0 \le \phi \le \frac{\pi}{3}\ rad$, and $0 \le z \le 1\ m$ by integration . Check your result without performing the integration.

Answer

a. By integration

$1 \le \rho \le 2$	$0 \le \phi \le \frac{\pi}{3}$	$0 \le z \le 1$

$$dv = \rho\ d\rho\ d\phi\ dz$$

$$\text{volume} = \iiint dv$$

$$V = \int_0^1 \int_0^{\frac{\pi}{3}} \int_1^2 \rho\ d\rho\ d\phi\ dz$$

$$V = \left[\frac{\rho^2}{2}\right]_1^2 \left[\phi\right]_0^{\frac{\pi}{3}} \left[z\right]_0^1$$

$$V = \frac{3}{2} * \frac{\pi}{3} * 1 = \frac{\pi}{2}\ m^3$$

b. Without integration

$$V = \frac{1}{6}[\text{Volume of outer cylinder} - \text{Volume of inner cylinder}]$$

$$V = \frac{1}{6}[\pi {\rho_2}^2 l - \pi {\rho_1}^2 l]$$

$$V = \frac{\pi}{6}[4 - 1] * 1 = \frac{\pi}{6} * 1 * 3 = \frac{\pi}{2}$$

2 Determine the area S of a surface in a spherical coordinate system as $r = 2\ m$ and $\frac{\pi}{4} \le \theta \le \frac{\pi}{3}$ rad.

Answer

$r = 2m$	$\frac{\pi}{4} \le \theta \le \frac{\pi}{3}$	$0 \le \phi \le 2\pi$

$$\text{surface area} = \iint ds$$

$$A = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} r^2 \sin\theta \ d\theta \ d\phi$$

$$A = (2)^2 * \left[-\cos\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left[\emptyset\right]_0^{2\pi}$$

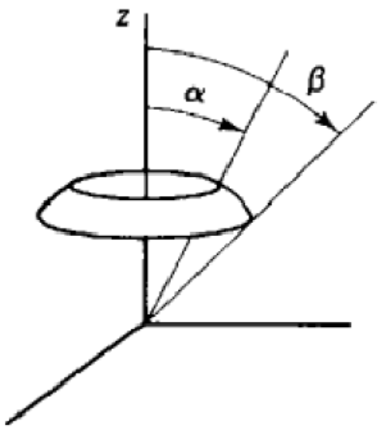
$$A = 4\left[\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\right] 2\pi$$

$$A = 8\pi\left[\frac{1}{\sqrt{2}} - \frac{1}{2}\right]$$

$$A = 5.205\ m^2$$

3 Use the spherical coordinate system to find the area of the strip $\alpha \leq \theta \leq \beta$ on the spherical shell of $r = a$ show this strip by sketching. What result when $\alpha = 0$ and $\beta = \pi$?

Answer



Strip $\alpha \leq \theta \leq \beta$ [spherical shell] $r = a$

$$A = \iint ds$$

$$A = \int_0^{2\pi} \int_\alpha^\beta r^2 \sin\theta \, d\theta \, d\phi$$

$$A = a^2 \Big[-\cos\theta\Big]_\alpha^\beta \Big[\phi\Big]_0^{2\pi}$$

$$A = a^2[cos\ \alpha - cos\ \beta\,]2\pi$$

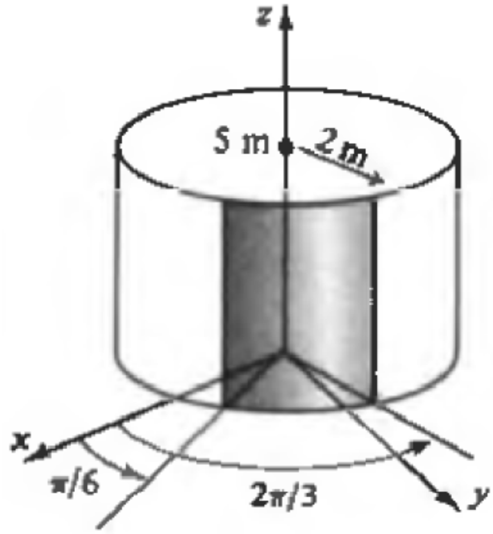
$$A = 2\pi a^2[cos\ \alpha - cos\ \beta\,]$$

For ($\alpha = 0$, $\beta = \pi$) , the strip will become a sphere

$$A = 2\pi a^2[\cos(0) - \cos(\pi)]$$

$$A = 4\pi a^2$$

4 Use the cylindrical coordinate system to find the area of the curved surface of a right circular cylinder where : $r = 2m$, $h = 5m$ and $30° \leq \phi \leq 120°$ as shown in fig



Answer

$$\rho = 2$$

$$\frac{\pi}{6} \leq \phi \leq \frac{2\pi}{3}$$

$$h = 5m$$

Area of the curved surface

i.e. $\rho = \text{constant} = 2$

$$ds = \rho \, d\phi \, dz$$

$$A = \rho \int_0^5 \int_{\frac{\pi}{6}}^{\frac{2}{3}\pi} d\phi \, dz$$

$$A = \rho \left[z \right]_0^5 \left[\phi \right]_{\frac{\pi}{6}}^{\frac{2\pi}{3}}$$

$$A = 2 \times 5 \times \left[\frac{2\pi}{3} - \frac{\pi}{6} \right] = 5\pi \, m^2$$

5 Given the point $P\ (5\text{m}, 60°, 2\text{m})$ and $Q\ (2\text{m}, 110°, -1\text{m})$

(a) Find the distance R_{PQ}

(b) Give a unit vector in Cartesian coordinates at P that is directed towards Q

Answer

$\text{P}(5,60°,2)$, $\text{Q}(2,110°,-1)$ [cylindrical]

(a) Find the distance R_{PQ}

Transforming points into Cartesian

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = z$$

Point P	Point Q
$x = 5 \cos 60° = 2.5$	$x = 2 \cos 110° = -0.684$
$y = 5 \sin 60° = 4.33$	$y = 2 \sin 110° = 1.879$
$z = 2$	$z = -1$
$P = (2.5\ , 4.33\ , 2)$	$Q = (-0.684\ , 1.879\ , -1)$

$$\bar{R}_{PQ} = (-0.684\ - 2.5)\bar{a}_x + (1.879\ - 4.33)\bar{a}_y + (-1\ - 2)\bar{a}_z$$

$$\bar{R}_{PQ} = -3.184\ \bar{a}_x - 2.45\ \bar{a}_y - 3\ \bar{a}_z$$

$$|\bar{R}_{PQ}| = \sqrt{(3.184)^2 + (2.45)^2 + (3)^2}$$

$$\boxed{|\bar{R}_{PQ}| = 5.014\ m}$$

(b) unit vector in Cartesian coordinates at P that is directed towards Q

$$\bar{a}_{PQ} = \frac{\bar{R}_{PQ}}{|\bar{R}_{PQ}|}$$

$$\bar{a}_{PQ} = \frac{-3.184\bar{a}_x - 2.45\bar{a}_y - 3\bar{a}_z}{5.014}$$

$$\boxed{\bar{a}_{PQ} = -0.635\bar{a}_x - 0.489\bar{a}_y - 0.598\bar{a}_z}$$